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QUASI–CLASSICAL APPROXIMATION TO THE QUANTUM PROPAGATOR OF A PERIODICALLY KICKED PARTICLE ON THE REAL LINE

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ABSTRACT

The quantum propagator of certain discrete–time area–preserving maps is considered. We evaluate the path–integral–like expression for this propagator within the quasi–classical approximation. The leading term is calculated by neglecting higher than second–order terms in an expansion of the action about the classical paths. The next–to–leading–order term is also obtained. We pay special attention to the possible appearance of Maslov–like phases. An estimate for the range of validity for the first–order quasi–classical approximation is also given.

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1. Introduction

The object of the present study is the quantum propagator of a dynamical system characterized by the time-dependent classical Hamiltonian

$$H(t) = \frac{p^2}{2m} + V(q) \sum_{n=-\infty}^{+\infty} \delta(n - t/\tau) , \quad (p,q) \in \mathbf{R}^2 , \quad \tau > 0 . \qquad (1)$$

It describes a point particle of mass m moving freely on the real line $\mathbf{R}$ for a time period τ followed by an instant kick generated by the potential $V(q)$. At the classical level the system (1) is equivalent to a discrete-time area-preserving map in phase space $\mathbf{R}^2$ and typically exhibits chaotic behavior.[1-4] At the quantum level this behavior should emerge in the quasi-classical approximation. Here we will perform this approximation to the quantum propagator in the q-representation. A previous study of such kind is due to Tabor[5] who has calculated the leading term of the quasi-classical approximation. For a treatment within the arena of phase space we refer to ref. 6.

After deriving the leading term we also perform an explicit calculation of the next-to-leading-order term. We particularly pay attention to the possible appearance of Maslov-like phases which have not been taken into account in the earlier calculation.[5] Using the result of the higher-order correction we obtain estimates for the range of validity of the quasi-classical approximation.

2. Quasi-Classical Approximation for the Quantum Propagator

Let us consider the following unitary operator on the Hilbert space $L^2(\mathbf{R})$ of square-integrable complex-valued functions defined on the real line $\mathbf{R}$

$$\hat{U} := \exp\left\{-\frac{\mathrm{i}}{\hbar} V(\hat{q})\,\tau\right\} \exp\left\{-\frac{\mathrm{i}}{\hbar}\frac{\hat{p}^2}{2m}\tau\right\} , \qquad (2)$$

where $\hat{q}$ and $\hat{p}$ are the usual position and momentum operators with commutation relation $\hat{q}\hat{p} - \hat{p}\hat{q} = \mathrm{i}\hbar$. The N-th power $\hat{U}^N$ decribes the time evolution of the quantum version of system (1) after N kicks.

In order to understand the time evolution one has to find a more explicit expression for $\hat{U}^N$. To this end, we will work in the q-representation where the propagator for N kicks takes the Feynman path-integral-like form

$$\begin{aligned} \langle x_N|\hat{U}^N|x_0\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}x_{N-1} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \, \langle x_N|\hat{U}|x_{N-1}\rangle \cdots \langle x_1|\hat{U}|x_0\rangle \\ &= \left(\frac{m}{2\pi\hbar\tau}\right)^{N/2} \mathrm{e}^{-\mathrm{i}N\pi/4} \int_{-\infty}^{+\infty} \mathrm{d}x_{N-1} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \, \exp\{(\mathrm{i}/\hbar) S_N(x_N, \ldots, x_0)\} . \end{aligned} \qquad (3)$$

Here the action of an arbitrary "path" $\{x_n\}_{n=0,\ldots,N}$ is defined as

$$S_N(x_N, \ldots, x_0) := \sum_{n=1}^{N} \left[\frac{m}{2\tau}(x_n - x_{n-1})^2 - V(x_n)\,\tau\right] . \qquad (4)$$

We will perform an approximate calculation of the "path integral" (3) by utilizing an approach often used in quasi-classical path-integral evaluations.[7,8] Namely, we will expand the action $S_N(x_N, \ldots, x_0)$ about a classical path $\{q_n^\alpha\}_{n=0,1,\ldots,N}$. Classical paths are those which start from $q_0^\alpha := x_0$, end in $q_N^\alpha := x_N$ and extremize the action (4). Note that for fixed boundary conditions there are in general more than one solution to the classical equation of motion $\partial S_N/\partial x_n = 0$. For this reason we have introduced the label α to enumerate these solutions.

For the expansion about the classical path with label α we set $x_n =: q_n^\alpha + \xi_n^\alpha$ with $\xi_N^\alpha = 0$, $\xi_0^\alpha = 0$. Expanding (4) up to second order in ξ and neglecting the higher–order terms we arrive at

$$S_N(x_N,\dots,x_0) \approx S_N(q_N^\alpha,\dots,q_0^\alpha) + \frac{1}{2}\sum_{n=1}^{N}\left[\frac{m}{\tau}(\xi_n^\alpha - \xi_{n-1}^\alpha)^2 - \tau\, V''(q_n^\alpha)(\xi_n^\alpha)^2\right]. \qquad (5)$$

Within this approximation the contribution of such a path to the propagator (3) is of the form $F_N^\alpha \exp\{(\mathrm{i}/\hbar) S_N(q_N^\alpha,\dots,q_0^\alpha)\}$ where we have introduced the amplitude

$$F_N^\alpha := \sqrt{\frac{m}{2\hbar\tau}}\,\frac{\mathrm{e}^{-\mathrm{i}N\pi/4}}{\pi^{N/2}} \int_{-\infty}^{+\infty} \mathrm{d}z_{N-1} \cdots \int_{-\infty}^{+\infty} \mathrm{d}z_1 \exp\left\{\mathrm{i}\sum_{n=1}^{N}\left[(z_n - z_{n-1})^2 - \frac{\tau^2}{m} V''(q_n^\alpha) z_n^2\right]\right\} \qquad (6)$$

with $z_n := \xi_n^\alpha \sqrt{(m/2\hbar\tau)}$. The integration is easily performed by diagonalizing the quadratic form in the exponent and using the Fresnel–integral formula

$$\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{\mathrm{i}\lambda z^2} = \frac{1}{\sqrt{|\lambda|}} \begin{cases} 1 & \text{for} \quad \lambda > 0 \\ \mathrm{e}^{-\mathrm{i}\pi/2} & \text{for} \quad \lambda < 0 \end{cases}. \qquad (7)$$

The resulting contribution to the propagator can be put into the form

$$\sqrt{\frac{m}{2\pi\hbar\tau\,|\det G_N^\alpha|}} \exp\left\{\frac{\mathrm{i}}{\hbar} S_N(q_N^\alpha,\dots,q_0^\alpha) - \mathrm{i}\frac{\pi}{2}\left(\nu_\alpha + \frac{1}{2}\right)\right\}, \qquad (8)$$

where G_N^α is the tridiagonal $(N-1)\times(N-1)$–matrix

$$G_N^\alpha := \begin{pmatrix} d_1^\alpha & -1 & 0 & \cdots & & 0 \\ -1 & d_2^\alpha & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -1 & d_{N-2}^\alpha & -1 \\ 0 & & \cdots & 0 & -1 & d_{N-1}^\alpha \end{pmatrix}, \quad d_n^\alpha := 2 - (\tau^2/m) V''(q_n^\alpha), \qquad (9)$$

and ν_α is a Maslov–like index defined as the number of negative eigenvalues of G_N^α. The complete propagator in the quasi–classical approximation is obtained from (8) by summing over all classical paths connecting q_0^α and q_N^α by N time steps τ:

$$\langle x_N|\hat{U}^N|x_0\rangle \approx \sum_\alpha \sqrt{\frac{m}{2\pi\hbar\tau|\det G_N^\alpha|}} \exp\left\{\frac{\mathrm{i}}{\hbar} S_N(q_N^\alpha,\dots,q_0^\alpha) - \mathrm{i}\frac{\pi}{2}\left(\nu_\alpha + \frac{1}{2}\right)\right\}. \qquad (10)$$

Note that for $N = 1$ the index ν_α is zero. However, as N is increasing the matrix G_N^α may develop negative eigenvalues.

The quasi–classical result (10) is similar to that of Tabor[5]. However, in ref. 5 the Maslov–like phases and the sum over all classical paths have been left out.

3. Higher–Order Corrections

In this section we will calculate the next–order corrections in $\hbar$ to the quasi–classical propagator. For this we have to take two more terms in the expansion (5) into account. With the substitution $z_n := \xi_n^\alpha \sqrt{(m/2\hbar\tau)}$ we arrive at

$$\begin{aligned} S_N(x_N,\dots,x_0) \approx\; & S_N(q_N^\alpha,\dots,q_0^\alpha) + \hbar \sum_{n=1}^{N}\left[(z_n - z_{n-1})^2 - \frac{\tau^2}{m} V''(q_n^\alpha) z_n^2\right] \\ & - \frac{\tau}{6}\left(\frac{2\hbar\tau}{m}\right)^{3/2} \sum_{n=1}^{N-1} V'''(q_n^\alpha) z_n^3 - \frac{\tau}{24}\left(\frac{2\hbar\tau}{m}\right)^2 \sum_{n=1}^{N-1} V^{(4)}(q_n^\alpha) z_n^4 \end{aligned} \qquad (11)$$

where $V'''(q) := \partial^3 V(q)/\partial q^3$ and $V^{(4)}(q) := \partial^4 V(q)/\partial q^4$. In the integral (6) there appear now extra factors in the exponential being of third and fourth order in the integration variables. Evalution of the resulting integral can be performed perturbatively in the standard way by utilizing the power series for the exponential with the higher–order terms in z. It will turn out that for the next order in $\hbar$ we have to expand terms of the form $\exp\{z^3\}$ up to second order in z^3. For the terms $\exp\{z^4\}$ it is sufficient to expand up to the first order. In doing so, the integration (6) is reduced to the calculation of "moments" of the form

$$\begin{aligned} \langle (z_n)^r (z_l)^s \rangle \;&:= \left(\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{\pi}}\right)^{N-1} \int \mathrm{d}^{N-1}\vec{z}\, \exp\left\{\mathrm{i}\vec{z}^{\mathsf{T}} G \vec{z}\right\} (z_n)^r (z_l)^s \\ &= \frac{\mathrm{e}^{-\mathrm{i}(\pi/2)\nu_\alpha}}{\sqrt{|\det G|}} \left(\frac{\partial}{\mathrm{i}\partial\eta_n}\right)^r \left(\frac{\partial}{\mathrm{i}\partial\eta_l}\right)^s \exp\left\{-\frac{\mathrm{i}}{4}\vec{\eta}^{\mathsf{T}} G^{-1} \vec{\eta}\right\}\Bigg|_{\vec{\eta}=\vec{0}}. \end{aligned} \qquad (12)$$

Here we have set $\vec{z} := (z_1,\dots,z_{N-1})^{\mathsf{T}}$ and $\vec{\eta} := (\eta_1,\dots,\eta_{N-1})^{\mathsf{T}}$. The superscript $^{\mathsf{T}}$ denotes transposition and for simplicity we have dropped the indices α and N of the matrix G defined in (9).

With the relations

$$\langle (z_n)^3 \rangle = 0\,, \quad \langle (z_n)^4 \rangle = -\frac{3}{4}\left(G^{-1}\right)^2_{nn}\,,$$
$$\langle (z_n)^3 (z_l)^3 \rangle = -\mathrm{i}\frac{3}{4}\left(G^{-1}\right)^3_{nl} - \mathrm{i}\frac{9}{8}(G^{-1})_{nn}(G^{-1})_{nl}(G^{-1})_{ll} \tag{13}$$

the next-order corrections in $\hbar$ to the amplitude (6) are obtained:

$$F_N^\alpha = \sqrt{\frac{m}{2\pi\hbar\tau|\det G_N^\alpha|}}\,\mathrm{e}^{-\mathrm{i}(\nu_\alpha+1/2)\pi/2}\left\{1 + \frac{\mathrm{i}\hbar\tau^3}{8m^2}\sum_{n=1}^{N-1} V^{(4)}(q_n^\alpha)(G^{-1})^2_{nn} \right.$$
$$\left. + \frac{\mathrm{i}\hbar\tau^5}{24m^3}\sum_{n,l=1}^{N-1} V'''(q_n^\alpha)V'''(q_l^\alpha)\left[2(G^{-1})^3_{nl} + 3(G^{-1})_{nn}(G^{-1})_{nl}(G^{-1})_{ll}\right]\right\}. \tag{14}$$

In the above $(G^{-1})_{nl}$ denotes the matrix elements of the inverse matrix of G.

These higher-order terms can now be used, for example, to roughly estimate the range of validity of the quasi-classical approximation (10). It can be argued[9] that for a path which is near the transition from regular to irregular motion that $|\det G| \sim N$. This leads to an estimate for diagonal elements, $|(G^{-1})_{nn}| \le N/4$. Assuming a similar bound also for the off-diagonal elements, the conditions for the validity of (10) are

$$\frac{\hbar\tau^3 N^3}{8(4m)^2}\sup_n\left|V^{(4)}(q_n^\alpha)\right| \ll 1 \quad\text{and}\quad \frac{5\hbar\tau^5 N^5}{24(4m)^3}\sup_n\left|V'''(q_n^\alpha)\right|^2 \ll 1\,. \tag{15}$$

These estimates lead to the following upper bounds for the number of iterations

$$N \ll \frac{\text{const.}}{\hbar^{1/3}}\,, \qquad N \ll \frac{\text{const.}}{\hbar^{1/5}}. \tag{16}$$

Such a scaling just before the transition from regular to irregular motion has also been found by Fishman, Grempel and Prange[10] in a renormalization-group approach to the kicked rotator.

Beyond this transition the contributions of the unstable paths dominate the sum in (10). For unstable paths it can be shown[9] that $|\det G| \sim \exp\{\lambda_\alpha N\}$ for large N where λ_α is the Lyapunov exponent of the unstable path with number α. Arguments similar to those made above lead to the bound

$$N \ll \frac{1}{\lambda_\alpha}\ln\left(\frac{\text{const.}}{\hbar}\right) \tag{17}$$

which is also in agreement with results obtain by other approaches.[4]

Let us note that recently it has been observed that the quasi-classical approximation for classically chaotic systems appears to be accurate for unexpectedly long times which are beyond the above bounds.[11] A crude explanation might go as follows. In the chaotic regime the number of classical paths to be taken into account in (10) increases exponentially with N. Hence, for large N the enormous number of classical paths together with their quadratic fluctuations in (10) may well represent the full path integral (3). But a better understanding is clearly necessary.

4. Conclusion

In this paper we have calculated the quasi-classical approximation to the unitary evolution operator of the form (2). In contrast to an earlier discussion[5] we have taken into account the possible appearance of Maslov-like phases. We have also calculated explicitly the next-to-leading orders in $\hbar$. These have been used to give an estimate for the validity of the quasi-classical approximation. Near the stochastic transition where regular paths contributing to (10) become irregular a scaling behavior for the maximum number of kicks can be found for which the approximation is valid. It is interesting to note that this behavior is similar to the one which has been found for the kicked rotator.[10] For further discussion, in particular on the properties of the determinant of G_N^α, we refer to ref. 9.

Finally, we point out that the present approach can be generalized to the kicked harmonic oscillator

$$H(t) = \frac{p^2}{2m} + \frac{1}{2}m\Omega^2 q^2 + V(q)\sum_{n=-\infty}^{+\infty}\delta(n - t/\tau), \quad (p,q) \in \mathbb{R}^2. \tag{18}$$

It turns out that for resonance, that is, $\Omega\tau$ is a non-zero integer multiple of π, the quantum propagator can be calculated exactly[12] for an arbitrary kicking potential V.

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